The importance of intention and order: teaching for conceptual understanding using handheld technology

Terry P. Vendlinski

National Center for Research on Evaluation,
Standards, and Student Testing (CRESST)
University of California, Los Angeles

April, 2009

This study was funded by Texas Instruments, Inc. The findings and opinions expressed herein are solely those of the author and do not necessarily reflect those of Texas Instruments. Correspondence concerning this study should be addressed to Terry Vendlinski, 300 Charles E. Young Dr., Suite 304, Los Angeles, CA 90095. Email: vendlins@ucla.edu. The author would like to thank Ms. Joan Morris, Ms. Cathy Michels, Dr. Jinok Kim, Dr. David Silver, and Dr. Kilchan Choi for their assistance with this study and the preparation of this report.

Abstract
While it seems clear that instruction on both procedures and concepts is important in mathematics education, the relative importance of each and the order teachers should use each to build instruction with handheld technology such as graphing calculators is still unsettled. To help begin answering these questions, we randomly divided 34 algebra students who were to begin their study of writing linear equations into two groups. One group received conceptual then procedural instruction on writing linear equations; the other group received the same instruction in reverse order. Both groups had access to the TI-Nspire™ handheld. Our findings suggest that conceptual instruction followed by instruction that integrates both concepts and procedures was the most efficacious in helping students learn to begin writing linear equations.

Background
Algebra has often been called the gateway course to higher education and economic success (Atanda, 1999). Algebra is given this distinction not only because it provides students fundamental math concepts upon which courses like the Calculus, Chemistry and Engineering will build (Driver, 1986; Harel, Selden, & Selden, 2006), but also because it begins to develop a process of abstract thinking in students that they will use to make sense of the world surround (Taylor, 1998). Accordingly, algebra can be seen as the language of generalization that allows categories of problems to be explained or solved by sets of common algorithms (Usiskin, 2004).

Algebra, however, is often not a *gateway* but a barrier to many students in the United States, especially students of color (Ball, 2003; Berkner & Chavez, 1997). In fact, the author is aware of school districts with failure rates of more than 90% in secondary school introductory algebra courses. Recent media coverage suggest that this is not anomalous (Helfand, 2006; Rubin, 2007), and a Summary of Results from the 2008 California Standards Test indicates the average passing rate for all Algebra student between 2001 and 2008 is about 22% (California Department of Education, 2008). This barrier seems to extend beyond the elementary and secondary school levels. A 2004 study of the nation's community college system suggests that only about 75% of students who initially enroll in elementary algebra are still enrolled at the deadline to withdraw and, of these students, over half ultimately fail the course. Students who are younger, male and of color all have a smaller chance of passing algebra, with young, males of color most at-risk (Meehan & Huntsman, 2004). Our latest attempts to improve math instruction at the secondary level have seemingly not improved these figures, despite large investments of resources. In fact, reports from colleges in California suggest that innumeracy among first year college students is growing (Kollars, 2008). The current state of affairs presents this nation a significant challenge.

Problems in Algebra Instruction

At least one of the reasons cited for the difficulty students often encounter when transitioning from arithmetic to algebra is that our instruction (both arithmetic and algebraic) seldom builds on the ways people actually learn (Bransford, Brown, & Cocking, 1999; Donovan & Bransford, 2005). In particular, teachers often do not build on students' prior pre-conceptions, understanding, intuition, or innate problem solving strategies, but often present math as a set of rules, procedures and facts to be memorized (Saxe, Gearhart, & Seltzer, 1999); mathematics is often presented in a seemingly random or disorganized manner; and procedural knowledge is often divorced from what the process or results *mean* (Fuson, Kalchman, & Bransford, 2005; Kilpatrick, Swafford, & Findell, 2001). Merely teaching *the* algorithms of algebra often does not seem to make their relationship to the reoccurring patterns in the academic and non-academic world clear to most students. On the contrary, such a teaching strategy is likely to obscure the thinking that underlies such algorithms (Bransford et al., 1999; Cooney & Wiegel, 2003; Donovan & Bransford, 2005; Donovan, Bransford, & Pellegrino, 1999; Kamii & Dominick, 1998). Nevertheless, a number of studies have suggested that the teaching, practicing and memorizing of algorithms occupies the majority of time in elementary and secondary math instruction in the United States and that such instruction has deleterious consequences. In particular, this type of instruction leads to math knowledge that is extremely context bound and not generalizable for most students (Clarke, Keitel, & Shimizu, 2006; National Mathematics Advisory Panel, 2008; NCES, 1999, 2001). As a results of this type of instruction, most K-12 mathematics students do not construct the meaning of core concepts and principles, cannot relate concepts to problem-solving skills and procedures, and view mathematics as a collection of isolated, meaningless procedures to be memorized, not understood (e.g., Carpenter & Lehrer, 1999; Hiebert & Carpenter, 1992; Porter, 1989; Schmidt et al., 2001; Stodolsky, 1988). Consequently, even when students have memorized algorithms, they may not be able to associate and apply *an* appropriate algorithm in particular situations or use algorithms as tools to solve problems.

Clearly, however, this is not meant to imply that students only need conceptual understanding. Procedures have developed as part of our culture (mathematics included) precisely because they allow for quick and accurate solutions and can reduce the cognitive load enough to allow for the representation and solution of increasingly complex problems (Saxe, 1999). Although the names vary, cognitive scientists generally suggest a distinction between knowing “what” and knowing “how to”. Declarative knowledge (the “what”) is encoded as schema, while procedural knowledge (the “how to”) is encoded as scripts (Schraw, 2006). In both cases, such encodings parallel the notion of “chunking” or encapsulating many steps into a single module (Miller, 1956) and help explain the development and power of expertise in a domain of knowledge (Chase & Simon, 1973). Recently, Harel, Selden and Selden (2006) have suggested that what binds the procedural and conceptual (they call it the operational and structural) together in mathematics is largely the symbolism of the domain, and that *understanding* these symbols in both a conceptual and procedural way is critical to understanding and being able to use mathematics such as algebra.

In this light, Kieran’s (2003) distinction between a *procedural* understanding and conceptual (she also calls it *structural*) understanding becomes more important. For Kieran, procedural understanding represents an ability to operate *on* numbers. In the algebra this sort of understanding would be demonstrated by knowing how to evaluate the expression $2x – 6y$ when $x = 1$ and $y = 2$ or how to verify $x = 2$ is the solution to the equation $3x + 1 = 7$. The procedural process uses formulas to generate a numerical answer. By contrast, *conceptual* understanding refers to the use of operations carried out on expressions or equations and obtaining an expression or equation (not a number) as a result. A key characteristic of this type of understanding is that the objects (including equations or expressions) are symbolic in nature and the operations are also symbolic, not computational. It is this conceptual understanding that over twenty years of research (see for example, Clement, Lockhead, & Monk, 1981; C. Kieran, 1984;

Rojano, 2002; Wagner, Rachlin, & Jensen, 1984), suggests is one of the key difficulties for students, and is a difficulty that manifests itself in a number of ways both inside and outside the mathematics classroom. For example, in algebra this becomes apparent in situations, often implicit, such as the distinction between an implied multiplication (2*x*) and an implied addition (2¼) or between a letter that represents an *unknown* and a letter that represents a *variable*. It is a procedural undertaking to confirm the value of *x* when *x* + -2 = 6, but quite another thing to write an equality to represent the relationship "there are six times as many teachers as students at this school" with variables (Clement et al., 1981). It is imperative that students understand such conceptual and procedural representations before real-world examples are introduced or the real-world examples can obscure mathematical understanding (Kaminski, Sloutsky, & Heckler, 2008).

Such a conceptual basis of mathematics (especially the concept of number and the mathematical operations on numbers) and how mathematics relates to real situations must be made clear if students are to be successful in algebra (Donovan & Bransford, 2005). In this sense, "the study of algebra begins as [students] observe how numbers form systems and as they generalize number patterns" (Kilpatrick et al., 2001, p. 20). Unfortunately, the use of such conceptual teaching to support thinking is often non-existent, especially in courses that come before algebra (Kieran, 2003).

Technology in Algebra Instruction

Technology has been seen as both a detractor from and as a means to teaching for deeper conceptual understanding. On the one hand, it is felt that the use of computers and calculators can hinder the development of computational ability and conceptual understanding in adolescents (California Department of Education, 2006; Wade, 2006). On the other hand there exists overwhelming evidence suggesting that children who have learned using calculators perform better than children without such experience (Groves, 1994; Stacey, 1994). In particular, studies have suggested that the use of graphing calculators can lead to better performance in algebra (Khoju, Miller, & Jaciw, 2005) and can have large positive effects on high-stakes, state-administered tests (Dimock & Sherron, 2005). Furthermore, studies have suggested that calculators have a great potential to develop "children's conceptual understanding and mental-computation strategies" if such instruction occurs prior to the formal teaching of relevant algorithms (Kamii & Dominick, 1998, p. 129).

Technology, in and of itself however, is not the panacea to the problem of unacceptable student mastery of the principles of algebra. In fact, an important finding in the study by Dimock and Sherron (2005) was that, "where teachers reported using graphing calculators to teach conceptual understanding … scores were 62 points lower [on state tests], holding all else constant." (p. 26). Two possible criticisms of such a finding were that the assessments used were only measuring recall, rather than conceptual understanding, of concepts, and that, despite their stated goals, teachers were actually teaching procedures, rather than intentionally teaching for deep conceptual understanding. We know from the work of Thompson, Philipp, Thompson, and Boyd (1994) that teacher instructional practices in mathematics are directly related to whether they perceive mathematics as a procedural (computational) or as a conceptual enterprise. Consequently, one of the suggestions the study by Dimock and Sherron made was to examine the use of the graphing calculator in contexts where the teacher's demonstrated goal was to improve deep conceptual understanding of students. In this sense, research suggests that, the ideas that teachers present must "have the potential for developing knowledge rich in relationships, [be] generative in the sense of being useful in understanding a range of interesting phenomena… [and must get] in touch with students' informal knowledge…" (Prawat, 1989, p. 33). Research also suggests that deep conceptual understanding is critical for developing long term proficiency in mathematics and even suggests that students should be taught conceptual understanding before learning algorithms (Kilpatrick et al., 2001). According to these authors, "the question, therefore, is not whether but how calculators should be used" (Kilpatrick et al., 2001, p.356).

These findings have important implications for the way we teach courses like introductory algebra. For example, when students are introduced to linear equations in introductory algebra, they are often taught that slope is "the change in y over the change in x," and they memorize the symbol (*m*) that represents slope. Many students (and teachers) mistakenly believe that this definition of slope is arbitrary, and a number of students erroneously quantify slope as the "change in x over the change in y" even after memorizing the definition of slope and demonstrating the ability to recall that definition on assessments (Beichner, 1994). Such instruction is very procedural in that it requires students to memorize a formula and definitions rather than to understand what the parts of the equation (e.g. slope) or the equation itself actually mean. We hypothesize that a deep conceptual understanding of slope (e.g. why it is defined the way it is, why this ratio is necessary in defining linear equations, etc.) taught with handheld graphing calculator technology and based on students pre-conceptions, intuition and innate problem-solving strategies would dramatically improve student understanding of and ability to describe two variable (independent and dependent) linear relationships. Moreover, we hypothesize that students who are instructed in a more conceptual way before being provided procedural algorithms would be more likely to correctly write linear equations from data tables than students who were instructed in the reverse order. If confirmed, such findings could have significant impacts for instruction with technology. First, these findings would suggest that it is possible to integrate technology and instruction in a way that will improve deep conceptual understanding. Second, they would provide some evidence of the importance of instructional sequence in such an integration. Finally, such findings could provide us insight into what, in the case of slope, that instructional sequence might be.

Research questions:

- Does the intentional use of graphing calculators to teach the idea of slope for deep conceptual understanding, improve a student's ability to write and describe a linear rule (equation) relating the independent and dependent variable?

- If graphing calculators are used to teach concepts for both procedural memorization and for conceptual understanding, does the order that lessons are taught produce significant differences in a student's ability to write and describe a linear rule (equation) relating the independent and dependent variable?

Methods

The sample for this study consisted of a single class of 34 eighth graders at a small, private K-8 school in Southern California. As a group, the students were slightly above the national norm reference group in their overall understanding of algebraic concepts (range 25 to 100 percent correct, with a mean score of 78% (SD = 17)), in their computational ability (range 13th to 99th percentile and a mean at the 65th percentile (SD = 22)), and in their ability to solve problems and interpret data (range 50th to 99th percentile, with a mean score at the 80th percentile) as measured by the Iowa Test of Basic Skills (ITBS). Within their cohort of eighth graders across the nation, this class was about eight (8) points above the national average percentile in their overall math ability. It should be noted, however, that this cohort took the ITBS for 8th grade in the fall of their eighth grade year. For the subjects of this study, the ITBS testing period occurred approximately one month before the study described in this paper began.

All eighth grade students at the school were enrolled in a full year introductory algebra course and these students were about to start their study of linear equations as our investigation began. In particular, the students were poised to learn how to write linear equations from different representations of data using the Texas Instruments TI-Nspire™ hand held computer (calculator) with the TI-84 faceplate. To ascertain the effects of teaching for conceptual understanding versus teaching for procedural fluency and the benefits, if any, resulting from how such teaching was ordered, the students were randomly divided into two groups. ITBS overall mathematics scores suggest that the groups were, in fact, equivalent in their overall understanding of algebraic concepts (Group 1 Mean = 75 percent, SD = 17; Group 2 Mean = 81 percent, SD = 17), in their problem-solving and data interpretation abilities (Group 1 Mean = 82nd percentile, SD = 15; Group 2 Mean = 77th percentile, SD = 16), and in their computational ability (Group 1 Mean = 67th percentile, SD = 21; Group 2 Mean = 63rd percentile, SD = 23). In none of these cases was the difference between the means significant ($t = -1.14$, $p = 0.26$; $t = 0.87$, $p = 0.39$; and $t = 0.53$, $p = 0.6$, respectively). Prior to instruction, the students were also asked to complete a pre-quiz of two questions in order to determine their ability to write linear equations before any instruction. The first question asked students to write the equation that would allow someone to find a particular ‘y’ value given a certain ‘x’ value. The second question asked students to explain the thinking they used to arrive at the equation. This quiz is at Appendix A1.

The first group (Group 1) received two days of instruction based on an intuitive approach to writing linear equations given the students prior knowledge. The discussion between the students and their teacher focused on student conceptual understanding of the meaning of the mathematical symbols in a linear equation that allows one to find an output value (y) given a certain input value (x). Tabular representations of the line were used so that students could easily model this representation in their handhelds. A detailed outline of this instruction is given in Appendix A2.

The second group of students (Group 2) received two days of instruction based on a procedural approach to writing the equation of a line. During this instruction, students learned the slope-intercept form of a linear equation ($y = mx + b$), were given a definition of slope (“rise over run”) and were shown how to find the slope and the y-intercept. Here again, both tabular and graphical representations of the line were used and students could easily model both representations in their handhelds. A detailed instructional plan is at Appendix A3.

After two days of instruction, the groups were again quizzed to assess their ability to write the equation of a line. The first question on this interim quiz, asked the students to determine an equation that could be used to find ‘y’ given an ‘x’ value for a given data set. The data was presented in two ways – in a “T” table and graphically on a Cartesian Plane – in order to accommodate both of the presentation styles available to students on the handheld and the pedagogical methods used by both instructors. The data on the first question of the interim quiz described an equation with a y-intercept of 0. The y-intercept was evident, but not labeled on the graphical representation of the data and the graphical representation clearly indicated the data was linear. The second question on the interim quiz again asked students to determine the equation that could be used to find ‘y’ given an ‘x’ value for a second data set. Data was again presented in both tabular and graphical form. In this case the y-intercept was -1 and, although not marked, was clearly represented on the graph. In both questions, the y-intercept, while not given in the table, could have been discovered by a student using patterns evident in the table to extrapolate a value. The last two questions of the quiz were designed to assess whether a student could accurately define “slope” (declarative knowledge) and whether a student could justify why “slope” had to be defined in this way (conceptual knowledge). All students, regardless of group, took the same version of this quiz at the same time and in the same classroom. The interim quiz, overlaid with the scoring rubric, can be found at Attachment A4.

Following the interim quiz, Group 1 received the two day instructional sequence based on the procedural approach to writing the equation of a line, and Group 2 received two days of instruction based on conceptual understanding.

Two teachers provided instruction during this study. The teacher, who taught the conceptual unit to both Groups 1 and 2, is a 15 year veteran teacher and has taught introductory algebra to eighth graders; pre-algebra, introductory algebra, geometry, and intermediate algebra at the high school level; and algebra in the community college. The teacher who taught both groups the procedural unit is a credentialed dual subject (math and science) teacher who has taught middle school mathematics for 12 years. The teachers took various measures to maximize the integrity of the study. Aside from discussing the general pedagogical approach with one another, the teachers did not share details of their instruction with one another until after the instruction was complete; they did not assign homework during this week of instruction; and, to minimize crosstalk or additional instruction, they asked students not to discuss concepts outside of class time.

At the end of the second instructional period, all students were asked to take a final quiz. This quiz was virtually identical to the interim quiz except that data values in both “T” tables and the accompanying graphs were changed. As a result, the slopes of both equations and the y-intercept of the second equation were different than the parallel item on the interim quiz. As was the case previously, all students took the same version of this quiz in the same classroom and at the same time. This quiz is shown at Appendix A5.

After the final quiz, the students were also asked to rate their satisfaction with the order of instruction they received (procedural first then conceptual or vice versa), their enjoyment of the handheld, how much they felt the handheld facilitated an understanding of writing the slope-intercept form of the equation, and how helpful they felt the handheld was in conceptually writing equations from the data (“T”) table. That questionnaire is at Appendix A6.

Instruction on linear equations (including instruction on the topics of relationships and functions; independent and dependent variables; and domain and range) continued during the next three weeks. This instruction integrated the conceptual and procedural aspects of writing linear equations. For example, instruction included the different representations of “change in x” (“run”, Δx, x2 – x1, difference between rows on the table, etc.), and how to “transform” equations so they could be compared (conceptual form, slope-intercept form, general form).

Students then took another quiz (called a “Graded Homework Review”) with a partner. The pairings for this quiz were made randomly at the beginning of the school year and did not consider study group membership. While the purpose of the Graded Homework review was formative, the results were used in this study as an indicator of emerging student proficiency in writing linear equations. This Graded Homework Review is at Appendix A7.

Finally, students took an individual test approximately one month after the pedagogical intervention described above in order to assess their cumulative learning and retention. The first item on this test asked that students write a rule that described the data given in an accompanying “T” table. The first page of that test is given in Appendix A8. Both the graded homework review item and test item had stakes attached (i.e. student results were used in grading) and were graded with a rubric for that purpose. For this study, however, the first question on each assessment was graded using two dichotomous measures. In both cases, the first outcome measure indicated whether the student had provided an equation with the correct slope, while the second measure was indicative of the correctness of the y-intercept. As students were not required to write equations in slope-intercept form, researchers transformed equations into slope-intercept form to assess y-intercepts, when necessary.

We used a simple Chi-square analysis and, where small expected values warranted, Fisher’s Exact Test to test the original hypothesis that conceptual instruction would result in better student ability to write linear equations than procedural instruction, and to test the effects of instructional order on student ability, explanations, and retention. In addition to these methods, we used descriptive statistics to determine whether students felt the intentional use of conceptual instruction with the handheld technology significantly improved their understanding over time.

The study will use a significance level of $\alpha = .05$ in order to determine whether to reject the null hypothesis associated with each of the above research questions. Specifically, these null hypotheses are:

- $H_{0\text{-}1}$: The intentional use of graphing calculators to teach the idea of slope for deep conceptual understanding, will not improve a student’s ability to write and describe a linear rule (equation) relating the independent and dependent variable compared to procedural instruction.
- $H_{0\text{-}2}$: If graphing calculators are used to teach concepts for both procedural memorization and for conceptual understanding, the order that lessons are taught will produce no differences in a student’s ability to write and describe a linear rule (equation) relating the independent and dependent variable.

The α = .05 level of significance is commonly used in educational (and other fields of) research as sufficient to mitigate Type-1 (false positive) errors. We also conducted a brief analysis of statistical power to mitigate possible Type-2 (false negative) errors using the Goodness-of-fit test for contingency tables (X^2) afforded by G* Power 3 software (Faul, Erdfelder, Lang, & Buchner, 2007). For this analysis we ran two tests – a "worst" case scenario, and an "expected" case scenario, where "worse" and "expected" are defined from the standpoint of how the procedural only or procedural-conceptual group would perform. From that point of view, the "worst" case is given by algebra passing rates commonly seen in districts the author has worked with around the state of California (10%); the "expected" case would roughly mirror performance on the California Standards Tests where only 41% of eighth graders taking algebra for the first time passed the test (California Department of Education, 2008). In this latter scenario, we used a 50% pass rate as that was more typical of the author's actual experience with eighth graders learning to write linear equations. In each case, we anticipated that 80% of the conceptual and 80% the conceptual-procedural group would succeed in writing linear equations.

For comparing dichotomous outcomes in two groups using the Chi-square statistic, the required sample size to achieve a power of .9 is 2 students in the worst case scenario (effect size ω = 2.33, df = 1) and is 30 students in the case of the expected outcome (effect size ω = .6, df = 1). Consequently, we conclude that the sample of 34 students available at the chosen school site should provide enough statistical power to avoid inappropriately accepting the null hypotheses.

Results

Of the 34 students in the class, 33 students took the pre-quiz. As was demonstrated by their pre-quiz scores, the randomized groups were roughly equivalent in their ability to write linear equations with a non-zero y-intercept (Question 1) and their ability to explain slope (Question 2). Fisher's Exact Test and Chi-square analysis confirmed that the abilities of these two groups, prior to instruction, were not significantly different on either pre-quiz question (Question 1: X^2 = .303, p = .582; Question 2: X^2 =2.265, p = .322). Fisher's Exact Test confirmed that pre-quiz results were independent of group (p = 1.00). See Table 1 and 2, respectively.

Table 1. Cross-tabulation of student results on Question 1 of the Pre-quiz by instructional group

	Group	
Response to Question 1	1	2
Incorrect Equation	15	15
Correct Equation	1	2

Table 2. Cross-tabulation of student results on Question 2 of the Pre-quiz by instructional group

Response to Question 2	Group 1	Group 2
1 – No answer or no recognition of a pattern in x or y mentioned	12	10
2 – Mentions pattern in x OR y but does not relate one to the other	4	5
3 – Relates the pattern in x AND y to each other (y increases half of x, etc.)	0	2

As seen in Table 3, however, after two days of instruction, every member of the group that received the more conceptually based curriculum (Group 1) who was originally unable to write the equation of a line in the pre-quiz was able to write a correct linear equation when the data was described by an equation with a y-intercept of zero. Only about half (53%) of such students in the group who received procedurally based instruction (Group 2) could do so.

Table 3. Interim quiz Question 1 results after first instructional set for students who were unable to write a correct linear equation on the pre-quiz

Response to Question 1	Group 1	Group 2
Incorrect Equation	0	7
Correct Equation	15	8

Although the Chi-square statistic is significant, we again computed significance using Fisher's Exact Test because half of the cells had expected values less than 5. Both tests suggest that the differences between groups are now significant ($X^2 = 9.13$; p = .003 and Fisher's 2 sided Significance = .006).

In a similar way, we conducted an analysis of the results from Question 2 of the interim quiz with students who were unable to write a linear equation on Question 1 of the pre-quiz. While the trend was similar in that the students taught conceptually were more able to correctly write a linear equation with a non-zero y-intercept, a chi-square analysis indicated that the difference between the two groups was not significant ($X^2 = 1.29$, p = .256) and Fisher's Exact Test (p = .45) confirmed that finding. See Table 4.

Table 4. Interim quiz Question 2 results after first instructional set for students who were unable to write a correct linear equation on the pre-quiz

	Group	
Response to Question 2	1	2
Incorrect Equation	8	11
Correct Equation	7	4

Each of the final questions on the interim quiz was analyzed to look for significant differences between students in Group 1 and Group 2. In both cases we controlled for pre-quiz results by eliminating students who had correctly written linear equations in the pre-quiz from our analyses. It should be noted, however, that all students who were able to write correct equations on the pre-quiz were also able to define slope (i.e. answer Question 3 of the interim quiz) correctly regardless of the group they were in. Although the small sample size prevented a meaningful statistical analysis, the cross-tabulation in Table 5 suggests little difference between the abilities of group receiving conceptual instruction (Group 1) and the group instructed in the explicit definition of slope (Group 2), and the difference was not significant ($X^2 = 6.014$, p = .111).

Table 5. Interim quiz Question 3 (define slope) results after first instructional set for students who were unable to write a correct linear equation on the pre-quiz

	Group	
Response to Question 3	1	2
0 – No answer or definition not related to slope	1	1
1 – Steepness, slant or change in x and y	4	9
2 – Change in x over change in y	4	0
3 – Change in y over change in x (rise over run, etc.)	6	5

Although our hypotheses only concerned the ability to write linear equations, it is interesting to note that the conceptual group did not seem able to define slope any better than the procedural group even though they were better at actually writing linear equations.

Because the sample size once again prevented a meaningful chi-square analysis with all four rubric points, we collapsed the scale shown in Table 6 into a two point version to determine if the distribution between score points indicative that students did not understand (score points 0, 1, and 2) and the score point that indicated the student could define slope (score point 3) were significantly different in the groups (see Table 6).

Table 6. Recode of interim quiz Question 3 results (Table 6) into two categories

Response to Question 3 (recoded)	Group 1	Group 2
Incorrect definition of slope	9	10
Correct definition of slope	6	5

Even after the score points were collapsed from four to two, the difference between the two groups was not significant (X^2 =.144, p = .705) and the two groups differed only slightly on their ability to define slope.

The final question of the interim quiz (Question 4) asked students to explain why slope had to be defined in the way they indicated in the previous question. Because of the small sample size, 75% of the cells in the chi-square analysis had expected values less than 5. To improve the quality of the analysis, we again collapsed the rubric score points into two. The higher of the two score points was given to students who gave some plausible explanation (e.g. one is finding y given x, or one is trying to determine how many rows down the "T" table to move, etc.). The lower category included explanations that were tautological or were incomprehensible. Although our analysis suggested that the differences between groups were not significant (X^2 = 3.33, p=.068), the ability to explain slope was evident only in the conceptual group (Group 1). Nevertheless, as seen in Table 7, this ability was still limited, even in Group 1. Also noteworthy was that none of the students who were able to correctly write linear equations on the pre-quiz were able to adequately explain why slope had to be defined the way they actually used it.

Table 7. Recode of interim quiz Question 4 results into two categories

Response to Question 4 (Recoded)	Group 1	Group 2
Unable to explain why slope was defined in the way they used it.	12	15
Able to explain why slope was defined in the way they used it.	3	0

We preformed analyses similar to those done on interim quiz for the first two questions on the final quiz. Here again, the ability level of the groups, each of which had now been taught using both a conceptual and a procedural method, was not significantly different on the first question of the final quiz (X^2 = .682, p = .409). There was, however, a difference between the performances of the first group (the group who received conceptual then procedural instruction) on Question 1 of the interim quiz and Question 1 of the final quiz. While all these students were able to correctly write a linear equation after conceptual instruction, only about two-thirds of them could still do so after then receiving the procedural instruction. Their counterparts receiving procedural instruction first (Group 2), however, were more fortunate. None of the students in this group who were able to write an equation on the interim quiz lost that ability after conceptual instruction. Moreover, over half the students who were unable to write a correct linear equation after receiving only procedural instruction, were able to write an equation correctly after the conceptual instruction. These trends can be seen in Tables 8 and 9 below.

Table 8. Cross-tabulation for Group 1 (Conceptual - Procedural) comparing results on the interim with results on the final quiz

	Response to Final Question 1	
Response to Interim Question 1	Incorrect	Correct
Incorrect	0	0
Correct	5	10

Table 9. Cross-tabulation for Group 2 (Procedural - Conceptual) comparing results on the interim with results on the final quiz

	Response to Final Question 1	
Response to Interim Question 1	Incorrect	Correct
Incorrect	3	4
Correct	0	8

Because all members of Group 1 were able to write a correct equation on Question 1 of the interim quiz, it was not possible to calculate either a chi-square or a Fisher's Exact Test statistic. For Group 2, however, the Chi-square statistic ($X^2 = 4.286$, $p = .038$) suggests that results on the first question on the final quiz are independent of the results on the interim quiz. As was the case before, the small number of expected observations in Table 9 adversely impacts the Chi-square statistic so we calculated Fisher's Exact Test statistic which indicates the results are not significant ($p = .077$).

As was the case with Question 2 on the interim quiz, there were no significant group differences between a student's results on Question 2 of the final quiz and the order in which the student received instruction ($X^2 = .144$, $p = .705$). See Table 10. Here again, all students who correctly wrote a linear equation on Question 1 of the pre-test also correctly wrote an equation for Question 2 of the final quiz so those results are not presented in Table 10.

Table 10. Cross-tabulation of final quiz Question 2 results after second instructional set for students who were UNABLE to write a correct linear equation on the pre-quiz

	Group	
Response to Question 2	1	2
Incorrect	5	6
Correct	10	9

Just as on the third question of the interim quiz, the results on Question 3 of the final quiz were not significantly different between the two groups; however, the quiz showed interesting trends. First, as seen in Table 11, the ability of all the students who were initially unable to write an equation improved in their ability to correctly define slope as measured by our collapsed two point rubric. The process of collapsing this data was exactly the same as the process described above for the interim quiz. A student's ability was independent of their group membership (X^2 = .136, p=.713). In the case of the students who had the ability to write equations for direct variations before the beginning of the study, the student who received conceptual instruction followed by procedural instruction failed to correctly define slope even though this student had been able to do so previously . The students, who received procedural instruction followed by conceptual instruction, did not suffer such "expertise reversal." While these results approach significance (X^2 = 3.00, p=.083), the number of subjects in this cohort is much too small to make this result statistically credible. That finding is confirmed by Fisher's Exact Test (p = .33). Rather, it is reported here as evidence of a developing trend suggested by the data. The outcomes for this question are presented in Tables 11 and 12, below.

Table 11. Cross-tabulation of the recode of final quiz Question 3 (define slope) results after second instructional set for students who were UNABLE to write a correct linear equation on the pre-quiz

	Group	
Response to Question 3 (recode)	1	2
Incorrect Definition of Slope	7	6
Correct Definition of Slope	8	9

Table 12. Cross-tabulation of the recode of final quiz Question 3 (define slope) results after second instructional set for students who were ABLE to write a correct linear equation on the pre-quiz

	Group	
Response to Question 3 (recode)	1	2
Incorrect definition of slope	1	0
Correct definition of slope	0	2

The results of our analysis of Question 4 of the final quiz largely reproduced the results we found for Question 4 of the interim quiz. In particular, we see in Table 13 that even after two days of conceptual instruction students have difficulty correctly explaining why slope must be defined as "change in y over change in x" (or "rise over run").

Table 13. Cross-tabulation of the final quiz Question 4 (explain the definition of slope) results after the second instructional set for students who were UNABLE to write a correct linear equation on the pre-quiz

	Group	
Response to Question 4 (Recode)	1	2
Unable to explain why slope was defined in the way they used it.	14	15
Able to explain why slope was defined in the way they used it.	1	0

Most importantly, students do not seem to realize that this definition relies on the assumption that x is defined to be the independent variable and y the dependent variable and relies on the assumption that x and y are in a relationship so that moving along the table (or graph) in one variable produces a proportional movement along the table (or graph) in the other variable. As has been observed above, the trend of students in Group 1 to lose a previously demonstrated ability is also evident in the data for this question. Specifically, some students who could correctly explain why slope had to be defined in a certain way after conceptual instruction, seemed to be unable to do so after procedural instruction. No changes were seen in the ability of students to explain the rationale for the definition of slope in the group that received procedural instruction followed by conceptual instruction, except in one instance. One of the students who had the ability to write linear equations before this study began and who received procedural instruction followed by conceptual instruction did improve in the ability to explain slope (see Table 14). As has been noted, due to the small numbers in each cohort, these results should only be used to suggest trends. They cannot be seen as statistically significant.

Table 14. Cross-tabulation of the final quiz Question 4 (explain the definition of slope) results after the second instructional set for students who were ABLE to write a correct linear equation on the pre-quiz

	Group	
Response to Question 4 (Recode)	1	2
Unable to explain why slope was defined in the way they used it.	1	1
Able to explain why slope was defined in the way they used it.	0	1

Finally, we looked at two other measures of student equation writing ability. The students took the first, Graded Homework Review 3, after a further three weeks of instruction on equation writing, relations, functions, domain, and range. The pedagogy during these three weeks integrated both conceptual instruction and procedural instruction. Students took this Graded Homework Review with a partner. All students in the class were able to write a linear equation with correct slope and, in all but three cases, the correct y-intercept. All the students who were able to correctly write an equation before instruction were able to do so after. The ability to write either part of the equation did not differ significantly between Group 1 and Group 2. All students correctly wrote slope on the Graded Homework Review and group membership was apparently not a significant factor for the three students who incorrectly wrote the y-intercept on that assessment ($X^2 = .37$, $p = .543$). This was confirmed by Fisher's Exact Test ($p = 1.00$).

We also analyzed the individual ability of students to write an equation on an individual summative test two weeks after the Graded Homework Review (five weeks after the end of initial instruction). Of the thirty students who were unable to correctly write a linear equation before any instruction (either procedural or conceptual), seventy percent of the class could write an entirely correct linear equation after instruction. More specifically, 77% of these students in the class could write a linear equation with correct slope and, surprisingly, 80% could write a linear equation with a correct y-intercept. While a Chi-square analysis suggests that the ability of these students to write the correct slope of an equation and their ability to write the correct y-intercept are not independent of one another ($X^2 = 7.87$; $p = .005$), it also suggests that there was no significant difference between students in Groups 1 and 2 for either of these aspects of writing an equation ($X^2 = .186$, p=.666 for slope; $X^2 = .833$, p=.361 for y-intercept). Given that 50% of the cells in the chi-square analysis have expected values less than five, we confirmed these results with Fisher's exact test (p = 1.00 for slope and p = .651 for y-intercept).

After the study, we asked students what about the order of instruction they received was most helpful and to rate their feelings about four statements concerning the use of the handheld on a seven point Likert scale. The actual student questionnaire is at Appendix A6.

Overall the students were divided in their preference for which type of instruction (conceptual or procedural) should come first. Many students thought the conceptual was "harder" and that seemed to have influenced their belief about which order was preferable. These students said things like, "It's easier to take a harder class and then come into an easy one," or "we get more understanding when we learn the hard stuff then easier." Other students wanted to "learn the basics … [then understand] why and how everything worked," or "[learn] the formula … [then get] a deeper comprehension, making it easy to understand and use."

The students were then asked for their reaction to four statements about the handheld computer. The first two statements concerned how much each student *enjoyed* using the handheld to learn to write linear equations procedurally and to write equations conceptually. While the median scores on these scales were comparable in that both groups "liked it" to some degree ($\text{Median}_{\text{procedural}} = 6$; $\text{Median}_{\text{conceptual}} = 6$), more students rated their enjoyment using the handheld to learn conceptually higher than they rated enjoyment using the handheld to learn procedurally ($\text{Mode}_{\text{procedural}} = 4$ ("didn't like or dislike it"); $\text{Mode}_{\text{conceptual}} = 7$ ("really liked it")).

The last two statements the students were asked to rate asked how much the handheld *helped* them understand how to write linear equations procedurally and to write equations conceptually. Here again, while the median scores on these scales were identical in that they found it "somewhat helpful" ($\text{Median}_{\text{procedural}} = 4$; $\text{Median}_{\text{conceptual}} = 4$), the modal scores were again different. The majority of students found the handhelds "somewhat helpful" in helping them understand the procedural approach to writing equations, and students were split on whether the handhelds were "somewhat helpful" or "very helpful" in helping them understand the conceptual approach to writing equations. These differences in how the two groups answered the survey questions were not statistically significant.

Discussion and Conclusions

We found that student outcomes differed after conceptual instruction and procedural instruction. In fact, even with our relatively small sample size and the limited duration of our study, the analysis of our data showed that the gains in student achievement at the interim quiz were significantly different between the groups, after controlling for student ability before instruction. Students taught to write linear equations conceptually, with instruction based on intuition and instruction that attempted to relate mathematical operations to their real world meaning, were significantly more able to write linear equations of a direct variation than were their counterparts taught with a more procedural method.

Whereas about half the students that received procedural instruction were able to correctly write a linear equation after two days of instruction, all of the students who received conceptually based instruction first were able to do so. While the same trend was also present for linear equations with a non-zero y-intercept, the differences between groups in that case were not significant after two days of instruction.

Given these results, it was interesting (and surprising) to note that the majority of students in both groups cold not correctly define slope. This is even more surprising given that all the students in the conceptual group and half the students taught procedurally could correctly write a direct variation. Clearly, in the case of the students taught conceptually, actually being able to define "slope" does not seem to be a critical component of correctly writing the equation of a line. Just as surprising, was the students' inability to explain why slope had to be defined a certain way. We felt that, for the students taught conceptually, such an ability would be a distinguishing characteristic of their knowledge. Evidently, this is not the case. While all of the students could write a direct variation that required the use of the concept of slope, most could not explain why slope had to be defined the way it was. We conclude that the ability to explain this concept may be more important for the teacher to intentionally teach slope in a conceptual manner than for students required to write a linear equation, if our only goal is for the student to be able to write a direct variation. This warrants more study but it seems likely that restricting our expectations in that way could adversely impact these students fully understanding "rates of change" and introductory calculus in the future. Based on the results then, the answer to our first question, "Does the intentional use of graphing calculators to teach the idea of slope for deep conceptual understanding, improve a student's ability to write and describe a linear rule (equation) relating the independent and dependent variable?" seems to be a clear "yes" in the case of direct variation and a much more qualified "yes" in the case of all linear equations.

We also found the order that concepts and procedures were presented was important. Conceptually based instruction was more likely to improve the performance of students who had previously been taught procedurally and had not mastered the content (4 out of 7 students). Moreover, such conceptual instruction did not seem to impede the students who had already mastered the ability to correctly write equations of direct variations. Every one of the students, who could correctly write a direct variation equation before instruction on the conceptual understanding of linear equations, could do so after conceptual instruction as well. The same was not the case for students who had previously demonstrated such an ability after conceptual instruction alone. When these previously successful students were quizzed after then receiving procedural instruction on writing such equations, about one-third of the group was unable to successfully complete the task they had effectively accomplished two days before. This suggests to us that "expert reversal" (Kalyuga, Ayres, Chandler, & Sweller, 2003) may be an important shortcoming of the conceptual – procedural order of instruction that seems absent from the procedural – conceptual pedagogical ordering. While our findings suggest the importance of conceptual instruction in both instances (as that instructional type seemed to routinely improve performance no matter where it was provided), we do not conclude that concepts should be taught after procedures merely to avoid the expert reversal effects. Rather, these findings suggest that concepts should be introduced before procedures and that, instead of introducing procedures with formal instruction, students should be encouraged to arrive at procedures to improve accuracy and speed. This is consistent with the findings of Rittle-Johnson and Star (2007). For example, during our *integrated* conceptual – procedural unit that followed this study (i.e. the time between the final quiz and the individual test) we asked the students how the various equations of a line provided by students might be easily compared. In part, such questioning is based on the work described by Rittle-Johnson and Star (2007) and the Cognitively Guided Instruction work of Carpenter and colleagues (Carpenter, Fennema, Levi, Franke, & Empson, 2000). Eventually, the students themselves suggested that the equation be written with two terms: a variable and its coefficient, and a constant. Although it could be argued that this is a result of actually teaching the slope-intercept form of the equation earlier, we feel it important to note that the students themselves concluded that the equation of a line should be written in this manner. The equation was not mandated but was, in the words of one student, “the simplest way to write the equation that everybody agrees on.”

In this case, the answer to our second question, “does the order that lessons are taught produce significant differences in a student’s ability to write and describe a linear rule (equation) relating the independent and dependent variable?” is less clear-cut. There were no significant differences between the groups after receiving both types of instruction; however, the conceptual group clearly regressed in a previously demonstrated ability after procedurally based instruction, while the procedural group clearly improved after conceptual instruction. While the groups, relative to one another, are statistically the same after the full course of instruction, the order of that instruction seems to have had a differential effect on the learning *progression* of each group. Conceptually based instruction seems to have consistently improved student ability, while procedural instruction did not.

In summary, while the differences between the groups’ abilities were significant at the interim quiz (suggesting that conceptual instruction should be preferred), the differences between the two groups at the end of the study were not significantly different. In other words, we must conclude that the two groups were essentially the same when they completed the entire conceptual – procedural sequence of instruction (regardless of order). Nevertheless, we must also conclude that when teaching the topic of writing equations for direct variations (and possibly even all linear equations), initially teaching the topic conceptually is likely to result in significantly better student learning than only teaching procedures. Moreover, we found that that when teaching procedurally then conceptually, conceptual instruction was able to make up for some, but not all, of the students’ learning deficits after procedural only instruction. We also found that procedural only instruction was not only likely to be less effective, but could also undo the learning of some students if provided after conceptual instruction. This was in contrast to some of the student’s opinions that it is better to learn the “basics” or “formula”, then focus on understanding. Our experience further suggests a conceptual approach followed by instruction that integrates concepts and procedures is most beneficial. Here again, this should be an area of further study.

As far as learning with technology, students’ feelings were similar whether learning procedurally or conceptually. Generally, they enjoyed using the handhelds and found them somewhat helpful in learning to write linear equations. More of the students liked using the handheld when learning conceptually than when learning procedurally and the students were just as apt to say they found the handheld “very helpful” as “somewhat helpful” when learning conceptually. When using the handheld to learn procedurally, most students said the handheld was only “somewhat helpful.

APPENDICES

APPENDIX A1
Pre-quiz

Linear Equations (Pre-quiz) Name:______________________________
November 2, 2007

This pre-quiz is intended to help us know what you understand about writing equations which allow us to calculate "y" when we know a specific value of "x". You should complete this pre-quiz without any outside help. You may use your calculator.

The equation 5x – 5 = y allows you to calculate the value for "y" when given a value for "x". For example, if you are told x = 1, you could substitute 1 for the x and write:

5 * **1** – 5 = y
5 – 5 = y
0 = y

If you had been told x = 2, you could write:

5 * **2** – 5 = y
10 – 5 = y
5 = y

We can represent these results in a table like the one to the right. The table shows that when x = 1, y = 0; and when x = 2, y = 5.

X	Y
1	**0**
2	**5**

Instead of finding the value of "y" when given "x" (like was done above), your task in this pre-quiz is to find **the equation** that would allow you to find a certain "y" value when given a certain "x" value.

1. Using words or symbols, show **one equation** that would give the value 4 for "y" when x = 2, 7 for "y" when x = 4 AND 10 for "y" when x = 6.

X	Y
2	4
4	7
6	10

2. Explain the steps or thinking you used to find (or to begin finding) the equation above. (You may continue on the back if necessary).

APPENDIX A2
Conceptual Instruction

Outline of instruction intended to develop deep conceptual understanding:

Day 1 –

Teacher presented a partial data set in a table:

X	Y
2	3
4	6
6	9

Students were asked to develop a verbal rule for the pattern (e.g. every time you move down a row, x changes by 2 and y changes by 3) individually, then to discuss in small groups of 2 or 3.

The teacher then asked students to make predictions about the values of y for other values of x (e.g. 8, 12, 20, 200 and 250). While the smaller values of x allowed students to confirm their rule, the larger values encouraged students to develop a more mathematical way to arrive at the value of y.

Students could write their data into the table of their handheld. The key idea here is that one can determine the "row" of the table for any entry by dividing by 2 (the change in x), and this row will be the same for a particular (x,y) pair since they are in a relationship. Multiplying the row by the change in y will tell you the y-value since the y-intercept is zero.

The teacher then asked if this works when you "skip" an entry (e.g. in the above activity the pattern in x goes 2, 4, 6, 8, 12). Students will generally decide that the value 10 "can" be included, but it need not be.

The teacher then asked how you might write a more general rule for any value of x. While the students often don't think about the "odd" values of x, these values will "work", but will be "half a row" in this table. This question led to an interesting discussion about whether if "half a row on x" is the same as "half a row on y", the pattern would "work."

Students entered their rule into the equation Y_1 and use the Table feature of the handheld to check the rule. The teacher then asked if (0,0) will work in the table? In their rule? This concept wasn't further explored at the time, but was asked to reinforce the idea that the table can include values above as well as values below the given entries, to prepare students to discuss domain and range of functions at a later date, and to begin the thinking that points that satisfy the equation must be in the table and points in the table must satisfy the equation that accurately represents the data set.

A final set of data (developed using the rule $y = 3/5\ x$ and x values of 5, 10, 15, 30, and 250) was presented to the students.

The final discussion centered on what dividing by "change in x" represents (the row number of the x entry) and what multiplying that by "change in y" represents (the y value of that row).

Day 2 –
Students explored data sets that don't contain the origin and discussed the necessity of considering the y-intercept. An initial data set was presented to the class:

X	Y
0	1
2	4
4	7
6	10

Students were again asked to develop a verbal rule for the pattern (e.g. every time you move down a row, x changes by 2 and y changes by 3), then discussed in small groups of 2 or 3. We wanted the students to see that, while you still add 3 to every preceding y value, you can't just multiply change in y by the row number to find y. In this case, that value must be added to 1.

As was done in Day 1, the teacher asked students to make predictions about the values of y for other values of x (e.g. 8, 12, 20, 200 and 250). This requires that students think about the relationships of x to y and not just one x to another x or one y to another y.

Students could enter data into the table of their handheld. The key idea here is that one can determine the "row" of the table for any entry by dividing by 2 (the change in x), and this row will be the same for a particular (x,y) pair since they are in a relationship. Multiplying the row by the change in y will tell you how much to add to the y-value associated with x = 0 (the y-intercept) to get the final y-value.

The teacher then asked how you might write a more general rule for any value of x. Students individually, and then in small groups arrived at an equation.

Students entered their rule into the equation Y_2 and used the Table feature of the calculator to check the rule. They made corrections to the equation as necessary.

The teacher then asked how the table today (X, Y_2) differs from the table yesterday (X, Y_1). This was followed by a discussion of how the rules today (for Y_2 and for Y_1) differed. The concept of the y-intercept (the value associated with x = 0) was discussed but not identified by name unless mentioned by the students.

A second set of data (using the rule $y = 3/5\,x + -2$ and x values of 5, 10, 15, 30, and 250) were presented to the students and the students were asked to develop the equation based on the data and to check this in the calculator. A discussion about what might be wrong with equations that didn't behave as expected ensued.

A final set of data like the one at the right, was explored:

X	Y
3	4
5	7
7	10

Students were asked to write an equation for this data. As with the case where the y-intercept was not zero, this data set required students to either estimate the y-intercept or to determine the “row” of that data pair. (Note that students could still determine the row by dividing by change in x, but that the row number wasn’t an integer value). Students estimated y values for x values of 9, 15, 35 and 101) to verify their rule.

The final discussion of the lesson focused on whether the same “strategy” (i.e. dividing x by change in x and multiplying by change in y, then adding to the y-intercept) is useful for writing equations for all lines and why.

APPENDIX A3
Procedural Instruction

Outline of instruction intended to develop procedural ability:

The lesson began when the teacher projected an axis graph of the coordinate plane on the whiteboard. The coordinate plane was discussed as being the intersection of two number lines and students labeled the x and y axes. The point of intersection was defined as the origin and student practiced naming points on the plane.

Students chose two plots on the coordinate plane and drew a line through the two points. Reference was made to a geometry unit from the seventh grade year where student defined a line as the connection of any two points. Students were asked to describe the line. Responses included things like: “it is straight”; “it goes up”; “It is an incline.” The teacher led the students to discover other points on the line by looking at the graph. These points were arranged in a data table.

The definition of slope was written on the board as “rise over run.” The students were told that this slope represented the steepness of a line. Rise was shown to be the vertical change on the y-axis, while run was shown to be a move along the x-axis. To find the rise and the run, two points were chosen and the teacher drew the change in x and y on the coordinate plane on the board. The line on the board had a rise of 3 and a run of 6. Students chose different points along the same line and determined the rise and the run of these points. Students discovered a relationship between the fractions and described them as “equal fractions and ratios.”

A different line (also positive slope) was drawn on the board and students demonstrated understanding by correctly naming the rise and the run.

The teacher drew points (-1, 2) and (3, -1) on the board. Student copied these on graph paper at their desks. The students, working with a partner (sitting next to them), were asked to create a line and determine the slope. The teacher led a discussion about how this line was different. What made it so? Could we use the same technique to find the rise and run? How would that be done? The students discovered that slope could go up or down. The teacher defined these slopes as positive and negative, respectively.

The teacher drew a new line (with positive slope) on the coordinate plane passing through points (-3, 3) and (1, 3). Different points were chosen (to determine the slope) and entered in a table of x and y values. The teacher asked the students to examine the table and explain what was happening to the x and y coordinates. They were asked to discuss their ideas with their partner. The students determined that there was a pattern and x and y changed in predictable increments. They demonstrated understanding by correctly predicting, then approaching the board and plotting, other coordinates on the line. Students could also do this in their handheld graphing calculators. The teacher explained that this change could be used to represent the rise over the run as slope or “m”. The “m” is used as the variable to represent slope and comes from the French (monter) for “to climb.”

Students were told they could find the slope of a line even when they could not see the line by using any two different coordinates and this formula. A data table was put on the board:

X	Y
1	2
2	4
3	6

Students were told they could find the slope by finding change in y (the rise) divided by change in x (the run). Sets of coordinates were chosen and used in the equation. Other sets of coordinates were chosen and shown to demonstrate that any set of points could be used to solve the equation successfully.

Students were given different tables with (x, y) coordinates and asked to find the slope of the line. These lines were drawn on the coordinate plane on the board and rise and run were demonstrated by the teacher moving a finger along the graph to visually show the rise and the run of the line. Here again, students could also add these tables and plot the points in the handheld.

The teacher drew the students' attention to the fact that all the lines cross the x and y axes at some particular point. These were defined to be the x-intercept and the y-intercept, respectively. Students were asked to determine the y-intercept of the line on the board. Other assorted lines were drawn and the exercise was repeated. Through discussion and extrapolation, the students stated that all y-intercepts have an x coordinate of zero. Students were taught how to name the y-intercept by only giving the y coordinate. A line was drawn though the origin. Students were asked to explain what was unique about this line (x and y intercepts of zero).

Four coordinate pairs were written in a data table:

X	Y
0	3
2	7
4	11
8	19

The students were asked to work with their partners to find the slope and the y-intercept without drawing the line. They worked only from the data table.

The teacher wrote the following equation on the board: $y = 2x + 3$ and told the students that this was an equation for a line. The class discussed how one could find the values for the variables by substituting constants for one variable and solving for the remaining unknown. The concept of dependent and independent variables was defined and explained. As a class, three values were chosen for x, the equation solved for y and the results entered in a data table. The teacher challenged the class to find the slope and the y-intercept. Working in cooperative groups the students drew on prior knowledge to complete the task. Students were asked to come to the board and draw the line. This activity was repeated with the equation $y = -2x +4$ and $y = \frac{3}{4}x + 1$.

The class was asked if they noticed anything special about the number that x was "attached to" (being multiplied by)? And what about the number that was being added? Another equation was demonstrated: $y = \frac{1}{2}x - 2$. It was determined that the coefficient of the variable was the slope of the line and the constant was the y-intercept. The teacher wrote the slope intercept formula on the board:

y = mx + b and told the students what the variables in the formula represented. The equation from the previous exercise was compared to the formula and different x values were substituted to solve the equation.

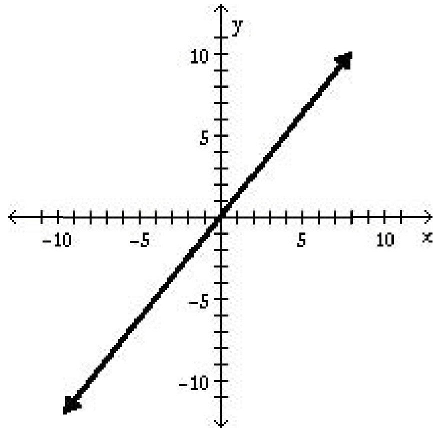
The teacher drew a line on the coordinate plane containing the following coordinates: (0,4), (2,0), and (5,6). The students were asked to work in pairs to discover the slope and y-intercept. Answers and solving techniques were discussed. Another line was drawn on the coordinate plane having the following coordinates (1,-7), (2, -14), and (3,-28). Students were again asked to find the slope and the y-intercept.

APPENDIX A4
Interim quiz overlaid with scoring rubric

Linear Equations (Interim-quiz) Name:______________________________
November 7, 2007

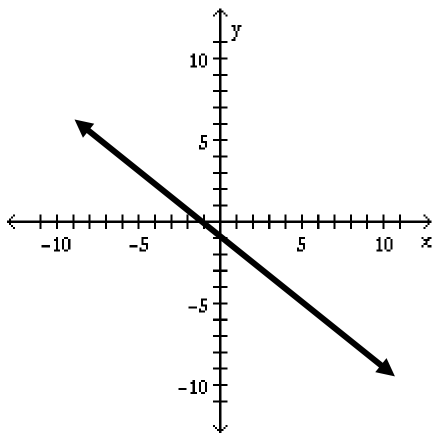
1. The <u>same</u> data is presented in two ways on the right and below (on a graph and in a table). Use either the graph or table to write an equation you can use to find "y" given any "x" value.

X	Y
4	**6**
6	**9**
8	**12**
x	**?**



2. The <u>same</u> data is presented in two ways on the right and below (on a graph and in a table). Use either the graph or table to write an equation you can use to find "y" given any "x" value.

X	Y
3	**-3**
6	**-5**
9	**-7**
x	**?**



3. What is the definition of "slope"?

0 – No answer or "I don't know"

1 – Steepness, slant or change in x and y

2 – Change in x over change in y

3 – Change in y over change in x (rise over run, etc.)

4. Why must slope be defined in this way?

0 – x is not related to y or is related using incorrect slope.

1 – That's just the way slope is defined or because it is Δy / Δx.

2 – Dividing by Δx tells you how far down the table you would be (e.g. row or how many groups down)

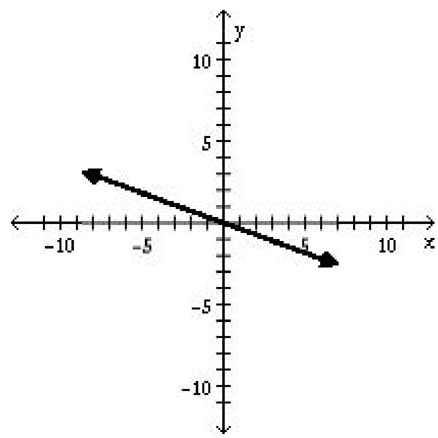
3 – When going from x to y or to find y when given x, you need to know how many groups you

APPENDIX A5
Final quiz

Linear Equations (Final quiz) Name:______________________________
November 9, 2007

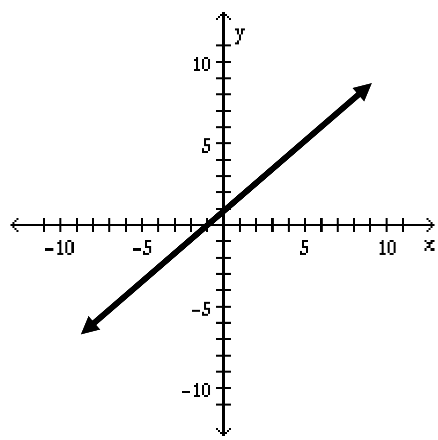
1. The same data is presented in two ways on the right and below (on a graph and in a table). Use either the graph or table to write an equation you can use to find "y" given any "x" value.

X	Y
6	**-2**
9	**-3**
12	**-4**
x	**?**



2. The same data is presented in two ways on the right and below (on a graph and in a table). Use either the graph or table to write an equation you can use to find "y" given any "x" value.

X	Y
-5	**-3**
0	**1**
5	**5**
x	**?**



3. What is the definition of "slope"?

4. Why must slope be defined in this way?

APPENDIX A6
Student Questionnaire

Student Questionnaire

Please do not write your name on this paper!

1. You were introduced to writing equations using two different methods. Which order of instruction did you receive on how to write equations?

[Teacher 2] / [Teacher 1] OR [Teacher 1] / [Teacher 2]*

*Note: Actual teacher names appeared in the [ ] on the actual questionnaire.

2. What made the order of teaching you chose above, the most helpful for you?

3. On the scale below, circle the number that represents how well you enjoyed using the graphing calculator in learning to write equations in the form y = mx + b.

Really   Didn't Like

4. On the scale below, circle the number that represents how well you enjoyed using the graphing calculator in learning to write equations fm the data in a table.

Really   Didn 't   Really

5. On the scale below, circle the number that represents how helpful the graphing calculator was in increasing your understanding of writing equations in the form y = mx + b.

Not Helpful   Somewhat Helpful   Very Helpful

6. On the scale below, circle the number that represents how helpful the graphing calculator was in increasing your understanding of writing equations from data in a table.

Not Helpful   Somewhat Helpful   Very Helpful

APPENDIX A7
Graded Homework Review

Algebra Name:________________________

Graded Homework Review 3 November 30, 2007

1. Show and explain how you would write a rule for any “y” given “x” if the pattern in the table on the right continues.

X	Y
2	**7**
5	**2**
8	**-3**

2. Graph the equation from problem number one on the graph paper provided.

3. Given the data in the table on the right, explain why you think:

a. The data would or would not represent a **relation**.

X	Y
2	**-4**
5	**-4**
7	**-4**

b. The data would or would not represent a **function**

4. What are the **range** and **domain** of the **function** (equation) $f(x) = x^2 + 5$? (Make sure you explain your answer and clearly identify which variable are used to define range and domain!)

5. Some people say that $-4^2 = (-4)^2$. Why would you agree or disagree with these people? (Be sure you fully explain your answer. Don’t just say they are equal or not equal but explain why you think your answer is correct).

APPENDIX A8
First Page of Individual Student Test

Algebra 1 Name:______________________________

Test Chapters 1- 2 December 12, 2007

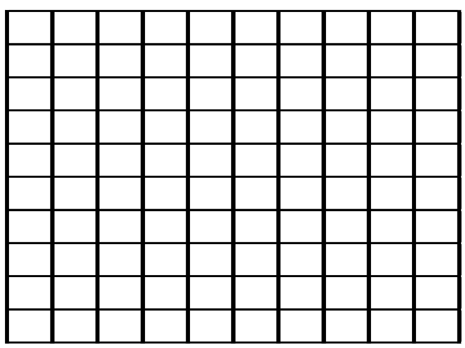
Show all your work. You may check with your calculator, but you must show me how you would solve each problem to receive credit.

1. Write a rule that describes the relation given in the following table:

X	Y
-4	5
2	-4
4	-7
8	-13

2. Would the ordered pair (-14, 20) fit the table given in problem 1 above? Explain your answer.

3. Graph the equation $y = |x - 2| + 2$. Make sure you label both axes and the increments you use.



4. What is the domain and range of the **equation** in problem 3? (Make sure you clearly specify which variable is associated with domain and which variable is associated with range in your answer).

References

Atanda, R. (1999). Do gatekeeper courses expand education options? *Education Statistics Quarterly, 1*(1).

Ball, D. L. (2003). *Mathematical Proficiency for All Students*. Santa Monica, CA: RAND.

Beichner, R. J. (1994). Testing student interpretation of kinematics graphs. *American Journal of Physics, 62*(8), 750 - 755.

Berkner, L., & Chavez, L. (1997). *Access to Postsecondary Education for the 1992 High School Graduates (NCES 98-105)*. Retrieved April 13, 2007, from http://nces.ed.gov/pubsearch/pubsinfo.asp?pubid=98105

Bransford, J. D., Brown, A. L., & Cocking, R. R. (Eds.). (1999). *How people learn: brain, mind, experience, and school*. Washington, DC.: National Academy Press.

California Department of Education (2008). *2008 Standardized Testing and Reporting (STAR) Program: summary of results.* Sacramento, CA: California Department of Education.

California Department of Education. (2006). *Mathematics Framework for California Public Schools Kindergarten Through Grade Twelve*. Sacramento, CA: California Department of Education.

Carpenter, T., & Lehrer, R. (1999). Teaching and learning mathematics with understanding. In E. Fennema & T. A. Romberg (Eds.), *Mathematics classrooms that promote understanding* (pp. 19 - 32). Mahwah, NJ: Erlbaum.

Carpenter, T. P., Fennema, E. E., Levi, L., Franke, M. L., & Empson, S. B. (2000). *Children's Mathematics: Cognitively Guided Instruction: A Guide for Workshop Leaders*. Washington, DC: The National Council of Teachers of Mathematics, Inc.

Chase, W. G., & Simon, H. A. (1973). Perception in chess. *Cognitive Psychology, 4*, 55 - 81.

Clarke, D., Keitel, C., & Shimizu, Y. (Eds.). (2006). *Mathematics Classrooms in Twelve Countries: the insiders perspective* (Vol. 1). Rotterdam, The Netherlands: Sense Publishers.

Clement, J., Lockhead, J., & Monk, G. (1981). Translation difficulties in learning mathematics. *American Mathematical Monthly, 88,* 286 - 290.

Cooney, T., & Wiegel, H. (2003). Examining the mathematics in mathematics teacher education. In A. J. B. M. A. C. C. K. J. K. F. K. S. Leung (Ed.), *Second international handbook pf mathematics education* (pp. 795-828). Dordrecht, The Netherlands: Kluwer.

Dimock, K. V., & Sherron, T. (2005). *Final Report of A Study of the Impact of Graphing Calculator Use on State Assessments*. Retrieved May 8, 2007, from http://education.ti.com/sites/US/downloads/pdf/graphing_use_st_assmnts_sedl_2005.pdf

Donovan, M. S., & Bransford, J. D. (Eds.). (2005). *How students learn: history, mathematics, and science in the classroom*. Washington D.C.: The National Academy Press.

Donovan, M. S., Bransford, J. D., & Pellegrino, J. W. (Eds.). (1999). *How people learn: bridging reserach and practice*. Washington, DC: National Academy Press.

Driver, R. (1986). *Why Math?* Berlin, Germany: Springer Verlag.

Faul, F., Erdfelder, E., Lang, A.-G., & Buchner, A. (2007). G*Power 3: A flexible statistical power analysis program for the social, behavioral, and biomedical sciences. *Behavior Research Methods, 39*, 175-191.

Fuson, K. C., Kalchman, M., & Bransford, J. D. (2005). Mathematical Understanding: An Introduction. In M. S. Donovan & J. D. Bransford (Eds.), *How students learn: history, mathematics, and science in the classroom*. Washington D.C.: The National Academy Press.

Groves, S. (1994). *The Effect of Calculator Use on third and Fourth Graders' Computation and Choice of Calculating Device.* Paper presented at the Eighteenth International Conference for the Psychology of Mathematics Education, Lisbon, Portugal.

Harel, G., Selden, A., & Selden, J. (2006). Advanced mathematical Thinking. In A. Gutierrez & P. Boero (Eds.), *Handbook of Research on the Psychology of Mathematics Education Past, Present and Future* (pp. 147 - 172). Rotterdam, Netherlands: Sense Publishers.

Helfand, D. (2006, January 30, 2006). A Formula for Failure in LA Schools. *Los Angeles Times*, pp. A1, A14-15.

Hiebert, J., & Carpenter, T. (1992). Learning and teaching with understanding. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 65 - 100). New York: Macmillan Publishing Company.

Kalyuga, S., Ayres, P., Chandler, P., & Sweller, J. (2003). Expertise reversal effect. *Educational Psychologist, 38*, 23-31.

Kamii, C., & Dominick, A. (1998). The Harmful Effects of Algorithms in Grades 1 - 4. In L. J. Morrow & M. J. Kenney (Eds.), *The Teaching and Learning of Algorithms in School Mathematics* (pp. 130 - 140). Washington DC: National Council of Teachers of Mathematics.

Kaminski, J. A., Sloutsky, V. M., & Heckler, A. F. (2008). The Advantage of Abstract Examples in Learning Math. *Science*, *320*, 454-455.

Khoju, M., Miller, G., & Jaciw, A. (2005). *Effectiveness of Graphing Calculators in K-12 Mathematics Achievement*. Retrieved May 8, 2007, from http://education.ti.com/sites/US/downloads/pdf/eei_graphingcalcreviewreport_2006.pdf

Kieran, C. (1984). A comparison between novice and more-expert algebra students on tasks dealing with the equivalence of equations. In J. M. Moser (Ed.), *Proceedings of the Sixth Annual Meeting of PME-NA* (pp. 83 - 91). Madison, WI: University of Wisconsin.

Kieran, C. (2003). The Learning and Teaching of School Algebra. In D. A. Grouws (Ed.), *Handbook on Research on Mathematics Teaching and Learning* (pp. 390 - 419). Reston, VA: National Council of Teachers of Mathematics.

Kilpatrick, J., Swafford, J., & Findell, B. (Eds.). (2001). *Adding + it up: Helping children learn mathematics*. Washington D.C.: National Academy Press.

Kollars, D. (2008, May 12). Despite high school algebra focus, more students need remedial college math. *Sacramento Bee,* p. A1.

Meehan, K., & Huntsman, H. (2004). *Pathways Through Algebra Project: Improving Student Success Rates in Elementary Algebra*. Retrieved April 11, 2007, from http://72.14.205.104/search?q=cache:z55AbRizN_AJ:www.ijournal.us/issue_09/ij_issue09_09_MeehanAndHuntsman_01.html+%22algebra%22+%26+success&hl=en&ct=clnk&cd=11&gl=us

Miller, G. A. (1956). The magical number seven, plus or minus two: some limits on our capacity for processing information. *The psychological review, 63*, 81 - 97.

National Mathematics Advisory Panel. (2008). *Foundations for Success: The Final Report of the National Mathematics Advisory Panel*. Washington DC: U.S. Department of Education.

NCES. (1999). TIMSS 1995. http://nces.ed.gov/timss/timss95: National Center for Educational Statistics (NCES).

NCES. (2001). *Data from the National Assessment of Educational Progress (NEAP)*. Washington, D.C.: National Center for Educational Statistics.

Porter, A. C. (1989). A curriculum out of balance: The case of elementary school mathematics. *Educational Researcher, 18*(5), 9-15.

Prawat, R. S. (1989). Promoting Access to Knowledge, Strategy, and Disposition in Students: A Research Synthesis. *Review of Educational Research, 59*(1), 1-41.

Rittle-Johnson, B., & Star, J. (2007). Does Comparing Solution Methods Facilitate Conceptual and Procedural Knowledge? An Experimental Study on Learning to Solve Equations. *Journal of Educational Psychology, 99*(3), 561-574.

Rojano, T. (2002). Mathematics Learning in the Junior Secondary School: Students' Access to Significant Mathematical Ideas. In L. D. English (Ed.), *Handbook of International Research in Mathematics Education* (pp. 143 - 163). Mahwah, NJ: Lawrence Earlbaum and Associates.

Rubin, J. (2007, July 11, 2007). Locke High's weary teachers face a hard multiple-choice test. *Los Angeles Times*.

Saxe, G. B. (1999). Cognition, development, and cultural practices. In E. Turiel (Ed.), *Culture and Development: New Directions in Child Psychology*. San Francisco: Jossey-Bass.

Saxe, G. B., Gearhart, M., & Seltzer, M. (1999). Relations between classroom practices and student learning in the domain of fractions. *Cognition and Instruction, 17,* 1 - 24.

Schmidt, W. H., McKnight, C. C., Houang, R. T., Wang, H., Wiley, D. E., Cogan, L. S., et al. (2001). *Why Schools Matter: A Cross-National Comparison of Curriculum and Learning*. San Francisco, CA: Jossey-Bass.

Schraw, G. (2006). Knowledge: Structures and Processes. In P. A. Alexander & P. H. Winne (Eds.), *Handbook of Educational Psychology* (2nd ed., pp. 245-263). Mahwah, NJ: Lawrence Erlbaum Associates.

Stacey, K. (1994). Arithmetic with a Calculator: What Do Children Need to Learn? In G. Bell, B. Wright, N. Leeson & J. Geake (Eds.), *Challenges in Mathematics Education: Constraints on Construction*. Lismore, New South Wales, Australia: Mathematics Education Research Group of Australia.

Stodolsky, S. S. (1988). *The subject matters: Classroom activity in math and social studies*. Chicago, IL: University of Chicago Press.

Taylor, J. E. (1998). The Importance of Workplace and Everyday Mathematics. In M. S. E. Board (Ed.), *High School Mathematics at Work: Essays and Examples for the Education of All Students* (pp. 30 - 34). Washington DC: National Academy Press.

Thompson, A. G., Philipp, R. A., Thompson, P. W., & Boyd, B. A. (1994). Calculational and conceptual orientations in teaching mathematics. In D. B. Aichele & A. R. Coxford (Eds.), *Professional Development for Teachers in Mathematics: Fifty-seventh Yearbook of the National Council of Teachers of Mathematics* (pp. 79 - 92). Reston, VA: National Council of Teachers of Mathematics.

Usiskin, Z. (2004). *A significant amount of algebra*. Retrieved April 13, 2007, from http://www.math.leidenuniv.nl/~naw/serie5/deel05/jun2004/pdf/usiskin.pdf

Wade, W. R. (2006). Sound Off! A Dialogue between Calculator and Algebra. *Mathematics Teacher, 99*(6), 391 - 393.

Wagner, S., Rachlin, S. L., & Jensen, R. J. (1984). *Algebra Learning Project: Final report*. Athens, GA: University of Georgia.